Heat Transfer Through a Homogeneous Isotropic Turbulent Field

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Because of the large scale of the motion responsible for mixing in turbulent fields, turbulent transport processes differ from molecular transport processes in that the mixing depends on the previous history of the diffusing material and turbulent fields are generally nonhomogeneous.

The effect of the time dependency of the diffusion process is examined for the case of heat transfer from a hot wall to a cold wall through a turbulently flowing fluid. The fluid is assumed to have a uniform velocity and the turbulence is assumed to be homogeneous and isotropic. The calculations are carried out by assuming a distribution of heat sources along the hot wall and of heat sinks along the cold wall. G. I. Taylor's theory of turbulent diffusion for a homogeneous isotropic field is used to describe the properties of these sources and sinks. These calculations are compared with temperature profiles obtained as a solution to Fick's Law using a constant diffusion coefficient. A marked difference between the two sets of curves is obtained in the vicinity of the wall and in the beginning of the heat exchange section.

A calculated profile on the basis of an idealized model of heat transfer in channel flow is compared with actual measurements made by Page, Corcoran, Schlinger, and Sage (*7*) at a distance far enough downstream so that the temperature profile had reached a steady condition.

Turbulent diffusion, like molecular diffusion, occurs from the random motion of material. However, unlike molecular diffusion, the scale of the motion is of the order of magnitude of the size of the container, and two distinct differences arise:

1. Turbulent fields are generally nonhomogeneous and the diffusion rates in different positions reflect variation in the turbulence characteristics.
2. The motion of a particular set of tagged particles will depend on their previous history, i.e., their time of entry into the field. Only at very long periods of time, when the motion of the particles does not resemble the motion they had upon entry into the field, will the diffusion be time independent.

Molecular transport of heat has been described by Fick's Law:

$$\frac{\partial T}{\partial t} = \nabla \cdot (\alpha \nabla T) \qquad (1)$$

$$\alpha = \text{thermal diffusivity} = \frac{k}{\rho C_p}$$

For a homogeneous medium the thermal diffusivity α will be constant and Equation (1) becomes

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T \qquad (2)$$

Turbulent transport of heat has been described by equations of the same type as Equations (1) and (2), a turbulent exchange coefficient being used in place of the molecular thermal diffusivity. Such an approach is at best empirical, and in general the exchange coefficient will have a different value at each point in the field. As pointed out above, the variation would exist even in the case of a homogeneous, isotropic field because the turbulent-diffusion process is time dependent.

Taylor (*1*) in 1935 described this time dependency for the case of a point source or sink located in an infinite, homogeneous, isotropic field. This method of description has been substantiated by experimental investigations (*1* to *4*). However, its application to more complex systems involving a distribution of sources and sinks has not been explored. In this paper G. I. Taylor's development will be applied to the prediction of temperature profiles in a fluid flowing turbulently between parallel hot and cold walls. The turbulence will be assumed to be homogeneous and isotropic, and the velocity profile of the fluid will be assumed uniform.

The paper has been written to emphasize the empirical nature of Equations (1) and (2) when applied to turbulent transfer, to examine the magnitude of the effect of the time dependency, and to suggest what might be the basis for a different method of describing turbulent transfer.

The calculations will be carried out by assuming that the fluid in flowing between the two walls is exposed to a number of instantaneous differential plane sources of heat at the hot wall and a number of instantaneous differential plane sinks of heat of the same strength as the sources at the cold wall. The temperature profile at any position may be determined by summing the contributions of all of these sources and sinks. Therefore, the key to these calculations is the description of a single instantaneous source or sink.

SINGLE INSTANTANEOUS SOURCE

An instantaneous source consists of the instantaneous appearance of a number of particles at a given point in a field at a given time. As time proceeds, these particles diffuse into the field. The distribution of the particles in the field at any time may be described in terms of their mean square displacement in the x direction from the place at which they originated, $\overline{X^2}$. For molecular diffusion the rate of change of this distribution with time is constant. The process possesses no measurable time dependence. According to a relation derived by Einstein (*5*) for Brownian motion, this rate of change bears a definite relation to the molecular diffusivity:

$$\frac{d\overline{X^2}}{dt} = 2D \qquad (3)$$

G. I. Taylor (*1*) established an analogous relation for a turbulent field:

$$\frac{d\overline{X^2}}{dt} = 2\overline{u^2} \int_0^t R \, d\epsilon \qquad (4)$$

where $\overline{u^2}$ = root mean square velocity in the x direction

$R = \frac{\overline{u_t u_\epsilon}}{\overline{u^2}}$ = correlation coefficient

$u_t u_\epsilon$ = product of the velocities of a particle at times t and ϵ

$\overline{u_t u_\epsilon}$ = average for a large number of particles

G. I. Taylor (*6*) suggested that the correlation coefficient R might have an exponential form, and recent experimental evidences (2 and 3) have supported his suggestion:

$$R = e^{-s/\tau} \quad (5)$$

$$\tau = \int_0^\infty R\, ds \quad (6)$$

$$s = (t - \epsilon)$$

The quantity τ is a scale factor which may be viewed as a measure of the lifetime of the oddities. If Equations (5) and (6) are substituted into Equation (4), the rate of change of $\overline{X^2}$ may be expressed in terms of two constants, $\overline{u^2}$ and τ, which characterize the turbulence:

$$1/2 \frac{d\overline{X^2}}{dt} = \overline{u^2}\tau(1 - e^{-t/\tau}) \quad (7)$$

At long periods this equation resembles that for molecular diffusion in that the rate of change approaches a constant value. This constant value will be referred to as the eddy diffusivity.

$$E = \overline{u^2}\tau \quad (8)$$

Equation (7) describes the magnitude of the spread of diffusing material. It does not describe how the material is distributed in the field. In treating this problem Frenkiel (4), on the basis of experimental evidences, assumed that $\overline{X^2}$ is described by a gaussian distribution. An equivalent assumption is made in this paper to describe the behavior of a single instantaneous source of heat or mass:

$$\frac{\partial T}{\partial t} = f(t) \frac{\partial T}{\partial x^2} \quad (9)$$

$$\frac{\partial C}{\partial t} = f(t) \frac{\partial C}{\partial x^2} \quad (10)$$

The diffusion coefficient $f(t)$ is a function of time. This functionality can be related to Equation (7) through the following considerations.

A solution to Equation (10) for the case of an instantaneous source of N particles in an infinite one-dimensional field is

$$C = \frac{N}{2\sqrt{\pi}\left[\int_0^t f\, d\theta\right]^{1/2}} \cdot \exp\left\{-\frac{x^2}{4\int_0^t f\, d\theta}\right\} \quad (11)$$

By substitution, the preceding expression can be seen to satisfy Equation (10). The expression also satisfies the physical boundary conditions imposed since

$$\int_{-\infty}^{+\infty} c\, dx = N \quad (12)$$

The probability that any of the diffusing particles will have a displacement between x and $x + dx$ is

$$P(x, x + dx) = \frac{C\, dx}{N} \quad (13)$$

$$P(x, x + dx)\, dx = \frac{dx}{2\sqrt{\pi}\left[\int_0^t f\, d\theta\right]^{1/2}} \cdot \exp\left\{-\frac{x^2}{4\int_0^t f\, d\theta}\right\} \quad (14)$$

The root mean square displacement of a large number of particles becomes

$$\overline{X^2} = \int_{-\infty}^{+\infty} x^2 P(x, x + dx)\, dx$$

$$= \int_{-\infty}^{+\infty} \frac{x^2\, dx}{2\sqrt{\pi}\left[\int_0^t f\, d\theta\right]^{1/2}} \cdot \exp\left\{-\frac{x^2}{4\int_0^t f\, d\theta}\right\} \quad (15)$$

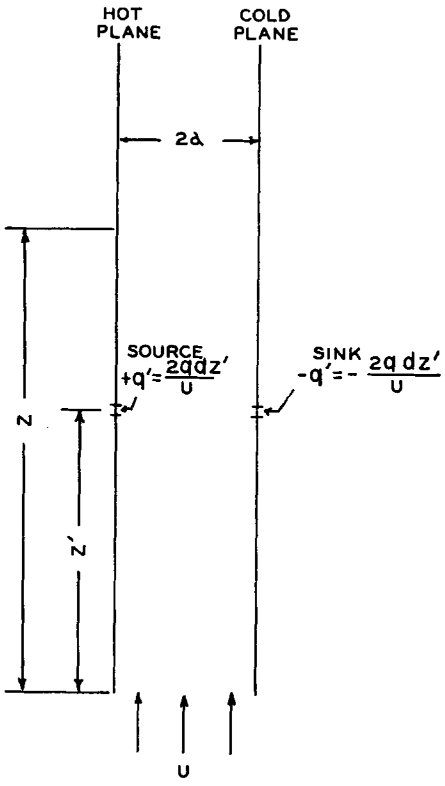

Fig. 1. Representation of the problem.

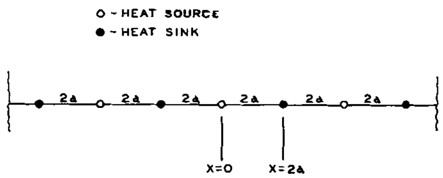

Fig. 2. Synthesis of solution for finite field from point-source solutions for infinite field.

Carrying out the preceding integration yields

$$\overline{X^2} = 2\int_0^t f\, d\theta$$

$$f = 1/2 \frac{d\overline{X^2}}{dt} \quad (16)$$

Substituting Equation (7) gives

$$f = \overline{u^2}\tau(1 - e^{-t/\tau}) \quad (17)$$

TEMPERATURE FIELD FROM AN INSTANTANEOUS SOURCE-SINK PAIR

Equations (9) and (17) may be used to describe the behavior of an instantaneous source or sink in a homogeneous isotropic field. The problem to be solved in this paper is illustrated in Figure 1. The fluid in flowing past Z' is subject over a distance dZ' to the addition of a small amount of heat at the hot wall and to the subtraction of a small amount of heat at the cold wall. As time proceeds, or as the fluid flows down the channel, the hot fluid from the source and the cold fluid from the sink diffuse away from the walls, causing an alteration of the temperature profile.

The effect of a single source-sink pair such as this, independent of the effect of the remainder of the heat transfer surface, will first be considered. If the observer is moving with the fluid with a velocity, U, changes in the Z direction become changes with respect to the time variable. The effect of this single source-sink pair may be represented as a solution to Equation (9). The boundary conditions for which the solution will be obtained are that at zero time a fluid of zero temperature is exposed to an instantaneous heat source of strength $+ q'$ at $x = 0$ and to an instantaneous heat sink of strength $-q'$ at $x = a$. At all other times no heat transfer occurs at the walls, and the temperature gradient at the walls is zero.

In order to obtain a solution of Equation (9) which will satisfy the boundary conditions, the method of images will be employed. This consists of finding the solution for a single source or sink in an infinite field and then distributing these sources and sinks in such a way that the boundary conditions of a finite field are satisfied.

The solution for an instantaneous source in an infinite field is

$$T = +\frac{q'}{\rho C_P 2\sqrt{\pi}\,[\phi(t)]^{1/2}} \cdot \exp\left\{-\frac{x^2}{4\phi(t)}\right\} \quad (18)$$

and for an instantaneous sink in an infinite field

$$T = -\frac{q'}{\rho C_P 2\sqrt{\pi}\,[\phi(t)]^{1/2}} \cdot \exp\left\{-\frac{x^2}{4\phi(t)}\right\} \quad (19)$$

$$\phi(t) = \int_0^t f\, d\theta \quad (20)$$

Substituting Equation (17) for f

$$\phi(t) = \int_0^{\overline{X^2}} 1/2 \frac{d\overline{X^2}}{d\theta}\, d\theta$$

$$= \int_0^t \overline{u^2}\tau(1 - e^{-\theta/\tau})\, d\theta$$

$$\phi(t) = 1/2\, \overline{X^2} = \overline{u^2}\tau t + \overline{u^2}\tau^2 e^{-t/\tau} - \overline{u^2}\tau^2$$

$$= E[t - \tau(1 - e^{-t/\tau})] \quad (21)$$

Equation (21) has been obtained by consideration of a point source. When Equation (21) is substituted into (19), an infinite temperature is obtained at the source point at zero time. To avoid this difficulty, a finite source with an initial distribution $\overline{X_0^2}$ will be assumed.* Then

$$\int_{\overline{X_0^2}}^{\overline{X^2}} 1/2 \frac{d\overline{X^2}}{d\theta}\, d\theta = \int_0^t \overline{u^2}\tau(1 - e^{-\theta/\tau})\, d\theta$$

$$\phi(t) = E[t - \tau(1 - e^{-t/\tau})] + 1/2\, \overline{X_0^2} \quad (22)$$

A solution for an instantaneous source-sink pair in a finite field can be synthesized by placing in an infinite field, as indicated in Figure 2, an infinite number of sources and sinks the behaviors of which are described by Equations (18), (19), and (22). By examination of Equations (18) and (19), it can be seen that if such source-sink pairs had been placed only at $x = 0$ and $x = 2a$ the expression obtained would give a finite temperature gradient at the two walls. This would violate the boundary condition of no heat transfer at the walls for times greater than zero.

Using the synthesis depicted in Figure 2, one obtains an infinite-series solution to Equation (9):

$$T = \frac{2q\, dt}{2\sqrt{\pi}\,\rho C_p[\phi(t)]^{1/2}} \cdot \left\{\cdots \exp\left(-\frac{(8a+x)^2}{4\phi(t)}\right) + \exp\left(-\frac{(4a+x)^2}{4\phi(t)}\right) + \exp\left(-\frac{x^2}{4\phi(t)}\right) + \exp\left(-\frac{(4a-x)^2}{4\phi(t)}\right) + \exp\left(-\frac{(8a-x)^2}{4\phi(t)}\right) + \cdots - \exp\left(-\frac{(6a+x)^2}{4\phi(t)}\right) - \exp\left(-\frac{(2a+x)^2}{4\phi(t)}\right) - \exp\left(-\frac{(2a-x)^2}{4\phi(t)}\right) - \exp\left(-\frac{(6a-x)^2}{4\phi(t)}\right) - \exp\left(-\frac{(10a-x)^2}{4\phi(t)}\right)\cdots\right\} \quad (23)$$

*Recent calculations indicate that the assumption of a finite source need not be made; however the contribution of molecular transfer must be included in the calculation of $\phi(t)$ in order that an infinite temperature not be obtained at the source location.

The substitution

$q' = 2q\, dt$

q = rate of heat transfer per unit area

was made in the preceding equation. The numeral 2 appears because in an infinite field heat will diffuse from both sides of the differential plane source.

SYNTHESIS OF SOURCE-SINK SOLUTIONS TO DESCRIBE CHANNEL HEAT TRANSFER

The calculation of the temperature profile at any value of Z (*see* Figure 1) can be accomplished by adding the contributions of all source-sink pairs at values of Z' between zero and Z. After substitution of

$$dt = \frac{dZ'}{U} \quad (24)$$

$$t = \frac{Z - Z'}{U} \quad (25)$$

into Equation (23), this summation may be expressed by the integral

$$T = \int_0^Z \frac{2q\dfrac{dZ'}{U}}{2\sqrt{\pi}\,\rho C_p\left[\phi\left(\dfrac{Z-Z'}{U}\right)\right]^{1/2}} \cdot \left[\cdots + \exp\left\{-\frac{x^2}{4\phi\left(\dfrac{Z-Z'}{U}\right)}\right\} + \cdots - \exp\left\{-\frac{(2a-x)^2}{4\phi\left(\dfrac{Z-Z'}{U}\right)}\right\}\cdots\right] \quad (26)$$

In the preceding equation Z' is the variable of integration and Z is constant.

It was not apparent how this integral could be evaluated analytically. Therefore in order to indicate the behavior of the solution a graphical integration was carried out by use of values of the parameters describing point-source diffusion which seemed reasonable on the basis of the limited number of data available in the literature. These were

$$\frac{\sqrt{\overline{u^2}}}{U} = 0.04$$

$$\frac{2aU}{E} = 275 \quad (27)$$

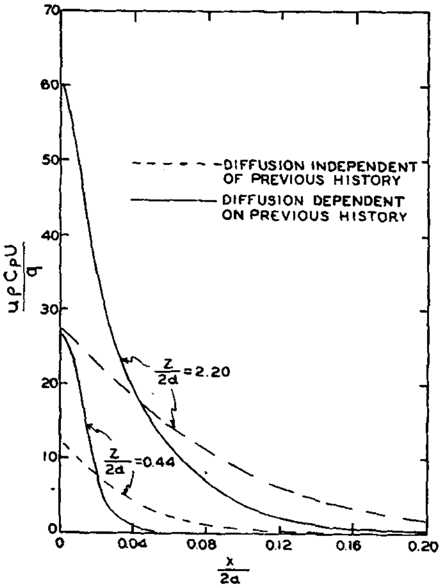

Fig. 3. Calculated temperature profiles for small Z/2a.

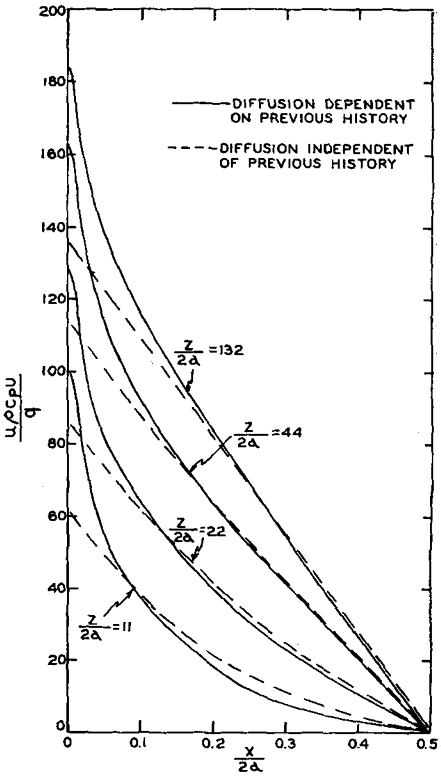

Fig. 4. Calculated temperature profiles for large Z/2a.

$$\frac{\overline{X_0^2}}{(2a)^2} = 10^{-4} \qquad (28)$$

The quantity q was assumed constant with respect to Z. The solutions obtained are presented in Figures 3 and 4. For the sake of demonstrating the time dependence of the diffusion process, solutions are also presented for the case of a constant diffusion coefficient of the magnitude presented in Equation (27).

Calculations were not carried out for $Z/2a > 132$ since very little change occurs in the temperature profile as Z is increased further. The value of $\overline{X_0^2}/2a$ chosen does not affect the shape of the curves for $x/2a > 0.01$ and $Z/2a \geq 11$. However for $Z/2a \leq 2.20$ the shape of the curves is very dependent upon $\overline{X_0^2}$ or upon the mechanism by which the fluid exchanges heat with the surface.

DISCUSSION OF RESULTS

The importance of the time dependence of the diffusion process is illustrated by the marked difference in the two calculated profiles at different values of Z. At small distances downstream of the beginning of the heat exchange section there is no resemblance between the temperature profile predicted for a homogeneous isotropic turbulent field and that calculated on the basis of a diffusion process independent of the previous history of the diffusing particles. At very large distances downstream the temperature profile in the center of the channel approximates what would exist if the diffusion coefficient were a constant. However, close to the wall there is a sharp rise in the temperature profile indicating a larger resistance to transfer in this region. Examination of Equation (7) shows that initially $\overline{X^2}$ varies with t^2 and that in the later stages of the process a linear dependence upon t is obtained. If the process were of the same type as described by Fick's Law, $\overline{X^2}$ would vary linearly with t throughout. The actual turbulent diffusion process exhibits a lower rate of transfer in its initial stages. Therefore, in regions where a large amount of the diffusing material has not been in the field for a long time a larger resistance to transfer will be in evidence. This accounts for the large temperature change in the vicinity of the wall and for the smaller penetration of heat in the beginning of the heat-exchange section for the actual turbulent transfer process as compared with one in which transfer is independent of previous history.

The magnitude of the effect of the time dependence will depend upon the parameter τ. For the particular turbulent field considered in Figures 3 and 4 the temperature profile at $Z/2a = 132$ starts to diverge from a straight line at $X/2a \backsimeq 0.15$. For a turbulent field of larger τ this divergence would begin to occur at a position closer to the center of the channel. The converse would be true for fields of smaller scale.

COMPARISON WITH EXPERIMENTAL DATA

The development in the preceding pages and the calculations presented in Figures 3 and 4 are based on the model of a homogeneous, isotropic field. Actual channel flow is neither homogeneous nor isotropic. Therefore, these calculations should be considered only as an illustration of the magnitude of the effect of the time dependence of turbulent diffusion. They should be at best a rough approximation of actual measurements. No data are available on the development of a temperature profile for heat exchange in channel flow. Measurements have been made by Page, Corcoran, Schlinger, and Sage (7) at distances far enough downstream that the temperature profile had reached a steady condition. These exhibited a form similar to that shown in Figure 4 for large $Z/2a$. In order to obtain a more direct comparison a temperature profile was calculated for test 40 of these experimenters by use of

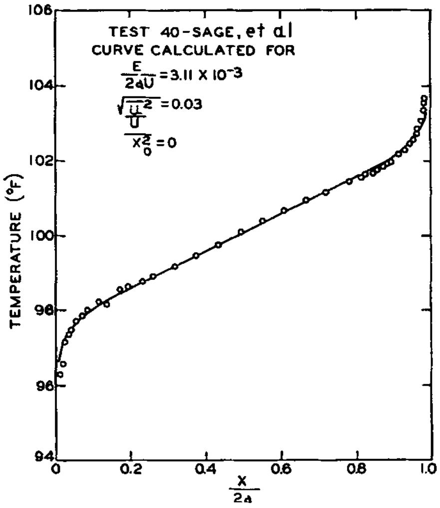

Fig. 5. **Comparison of calculated curve with experimental data.**

$$\frac{\sqrt{\overline{u^2}}}{U} = 0.03$$

$$\overline{X_0^2} = 0$$

The value assumed for the turbulence intensity seemed reasonable on the basis of measurements made with the hot-wire anemometer (8). Calculations were not made for values of $X/2a < 0.01$; therefore the assumed value for X_0^2 should not have much influence upon the calculated results. The value of the eddy diffusivity E used was obtained from the slope of the experimental temperature profile at the center of the channel.

$$\rho C_p E = \frac{q}{dt/dy}$$

The rate of heat transfer along the heated surface q was assumed constant. Under the conditions of the experiment q probably varied in an exponential manner, approaching a constant value at large Z.

The conditions of test 40 were such that if the diffusion coefficient were constant a linear temperature profile would result. The comparison of the calculated curve with the experimental data is shown in Figure 5. The fact that the calculated profile agrees with the data in the center of the channel is not surprising, as the value of E was selected so that this had to be the case. However, the agreement of the trend of the calculated curve and the data near the wall indicates that increased resistance near the boundary may not be due entirely to nonhomogeneities but may in a large part be due to the time dependence of the process.

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NOTATION

a = half width of the channel
C_P = heat capacity
D = molecular-diffusion coefficient
E = eddy diffusivity, value of f at infinite time
f = diffusion coefficient
k = thermal conductivity
q = rate of heat transfer per unit area
q' = instantaneous heat pulse
R = correlation coefficient
t = time
T = temperature
u = temperature difference from some base temperature (center of the channel for the case of channel flow)
$\sqrt{\overline{u^2}}$ = root-mean-square velocity of the diffusing material
U = average fluid velocity
x = distance from the wall
$\overline{X^2}$ = root-mean-square displacement of the diffusing particles
Z = distance downstream from the beginning of the heat-exchange section
Z' = location of a heat source or sink
α = thermal diffusivity
θ = time variable used in integration
τ = time scale characterizing turbulence = $\int_0^\infty R\, d\theta$
$\phi(t)$ = $\int_0^t f\, d\theta$

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